

CHARACTERIZATIONS OF THE GAMMA DISTRIBUTION BY INDEPENDENCE PROPERTY OF RANDOM VARIABLES

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ABSTRACT. Let $\{X_i, 1 \leq i \leq n\}$ be a sequence of i.i.d. sequence of positive random variables with common absolutely continuous cumulative distribution function $F(x)$ and probability density function $f(x)$ and $E(X^2) < \infty$. The random variables $X + Y$ and $\frac{(X-Y)^2}{(X+Y)^2}$ are independent if and only if X and Y have gamma distributions. In addition, the random variables S_n and $\frac{\sum_{i=1}^m (X_i)^2}{(S_n)^2}$ with $S_n = \sum_{i=1}^n X_i$ are independent for $1 \leq m < n$ if and only if X_i has gamma distribution for $i = 1, \dots, n$.

1. Introduction

Let $\{X_i, 1 \leq i \leq n\}$ be independent and identically distributed(i.i.d.) non-degenerate and positive random variables with common absolutely continuous cumulative distribution function $F(x)$.

Let X and Y be two independent non-degenerate positive random variables. It is known that X/Y and $X + Y$ are independent if and only if X and Y are gamma distributions with the same scale parameter as used in Lukacs(1955). By using the moment, Findeisen(1978) characterized the gamma distribution. Also, Hwang and Hu(1999) also proved a characterization of the gamma distribution by the independence of the sample mean and the sample coefficient of variation. Recently, Lee and Lim(2009) presented characterizations of gamma distribution with the

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property that the random variables $\frac{X_i X_j}{(\sum_{k=1}^n X_k)^2}$ and $\sum_{k=1}^n X_k$ are independent for $1 \leq i < j \leq n$ if and only if X_i has gamma distribution for $i = 1, \dots, n$.

In this paper, we obtain the characterizations of the gamma distribution by independence property of the quotient of sum and difference of random variables.

2. Results

THEOREM 2.1. *Let X and Y be nondegenerate and positive i.i.d. random variables with common absolutely continuous cumulative distribution function $F(X)$ and $E(X^2) < \infty$. The random variables $X + Y$ and $\frac{(X - Y)^2}{(X + Y)^2}$ are independent if and only if X and Y have gamma distributions.*

THEOREM 2.2. *Let $\{X_i, 1 \leq i \leq n\}$ be nondegenerate and positive i.i.d. random variables with common absolutely continuous cumulative distribution function $F(X)$ and $E(X^2) < \infty$. The random variables S_n and $\frac{\sum_{i=1}^m (X_i)^2}{(S_n)^2}$ are independent for $1 \leq m < n$, where $S_n = \sum_{i=1}^n X_i$ if and only if X_i has gamma distribution for $i = 1, \dots, n$.*

3. Proofs

Proof of Theorem 2.1. Since $\frac{(X - Y)^2}{(X + Y)^2}$ is a scale-invariant statistic, $\frac{(X - Y)^2}{(X + Y)^2}$ and $X + Y$ are independent [see Lukacs and Laha(1963)].

We have to prove the reverse. We denote the characteristic functions of $X + Y$, $\frac{(X - Y)^2}{(X + Y)^2}$ and $\left(X + Y, \frac{(X - Y)^2}{(X + Y)^2}\right)$ by $\phi_1(t)$, $\phi_2(s)$ and $\phi(t, s)$, respectively. The independence of $X + Y$ and $\frac{(X - Y)^2}{(X + Y)^2}$ is equivalent to

$$(3.1) \quad \phi(t, s) = \phi_1(t) \cdot \phi_2(s).$$

The left hand side of (3.1) becomes

$$\phi(t, s) = \int_0^\infty \int_0^\infty \exp\left(it(x + y) + \frac{is(x - y)^2}{(x + y)^2}\right) dF(x) dF(y).$$

Also the right hand side of (3.1) becomes

$$\begin{aligned}\phi_1(t) \cdot \phi_2(s) &= \int_0^\infty \int_0^\infty \exp\left(it(x+y)\right) dF(x)dF(y) \\ &\cdot \int_0^\infty \int_0^\infty \exp\left(\frac{is(x-y)^2}{(x+y)^2}\right) dF(x)dF(y).\end{aligned}$$

Then (3.1) gives

$$\begin{aligned}(3.2) \quad & \int_0^\infty \int_0^\infty \exp\left(it(x+y) + \frac{is(x-y)^2}{(x+y)^2}\right) dF(x)dF(y) \\ &= \int_0^\infty \int_0^\infty \exp\left(it(x+y)\right) dF(x)dF(y) \\ &\cdot \int_0^\infty \int_0^\infty \exp\left(\frac{is(x-y)^2}{(x+y)^2}\right) dF(x)dF(y).\end{aligned}$$

The integrals in (3.2) exist not only for reals t and s but also for complex values $t = u + iv$, $s = u^* + iv^*$, where u and u^* are reals, for which $v = \text{Im}(t) \geq 0$, $v^* = \text{Im}(s) \geq 0$ and they are analytic for all t, s for $v = \text{Im}(t) > 0$, $v^* = \text{Im}(s) > 0$, [see, Lukacs(1955)].

Differentiating (3.2) two times with respect to t and then one time respect to s and setting $s = 0$, we get

$$\begin{aligned}(3.3) \quad & \int_0^\infty \int_0^\infty (x-y)^2 \exp\left(it(x+y)\right) dF(x)dF(y) \\ &= \theta \int_0^\infty \int_0^\infty (x+y)^2 \exp\left(it(x+y)\right) dF(x)dF(y)\end{aligned}$$

where

$$\theta = E\left[\left(\frac{X-Y}{X+Y}\right)^2\right].$$

The random variable θ is bounded and the moments exists. Then we know that

$$(3.4) \quad \theta = E\left[\left(\frac{X-Y}{X+Y}\right)^2\right] = E\left[1 - \frac{2}{1 + \frac{X^2+Y^2}{2XY}}\right].$$

Note that, for $x > 0$ and $y > 0$, $0 < 2xy \leq x^2 + y^2$ and the equality on the right hand side occurs only if $x = y$. By the assumed continuity of $F(x)$, we find $P(x = y) = 0$ and $1 < \frac{x^2+y^2}{2xy} < \infty$. It follows $0 < \theta < 1$ by (3.4) .

Let $\varphi(t)$ be the characteristic function of $F(x)$. Then

$$\varphi'(t) = i \int_0^\infty x \exp(itx) dF(x), \quad \varphi''(t) = - \int_0^\infty x^2 \exp(itx) dF(x).$$

Expressing (3.3) as a differential equation for the characteristic function $\varphi(t)$, we get

$$\varphi''(t)\varphi(t) - 2(\varphi'(t))^2 + \varphi(t)\varphi''(t) = \theta[\varphi''(t)\varphi(t) + 2(\varphi'(t))^2 + \varphi(t)\varphi''(t)].$$

That is,

$$\frac{\varphi''(t)}{\varphi'(t)} = \left(\frac{1+\theta}{1-\theta} \right) \frac{\varphi'(t)}{\varphi(t)}, \quad 0 < \theta < 1.$$

After integrating with the initial conditions $\varphi(0) = 1, \varphi'(0) = iE(X)$, we get

$$(3.5) \quad \varphi'(t) = iE(X)(\varphi(t))^{\frac{1+\theta}{1-\theta}}, \quad \frac{1+\theta}{1-\theta} > 1.$$

The solution of this differential equation (3.5) with the above initial conditions is

$$\varphi(t) = \left(1 - \frac{iE(X)}{\lambda}t\right)^{-\lambda}, \quad \lambda = \frac{1-\theta}{2\theta} > 0.$$

Consequently, $F(x)$ is a gamma distribution. \square

Proof of Theorem 2.2. Since $\frac{\sum_{i=1}^m (X_i)^2}{(S_n)^2}$ is a scale-invariant statistic, $\frac{\sum_{i=1}^m (X_i)^2}{(S_n)^2}$ and S_n are independent [see Lukacs and Laha(1963)].

We have to prove the reverse. We denote the characteristic functions of $S_n, \frac{\sum_{i=1}^m (X_i)^2}{(S_n)^2}$ and $\left(S_n, \frac{\sum_{i=1}^m (X_i)^2}{(S_n)^2}\right)$ by $\phi_1(t), \phi_2(s)$ and $\phi(t, s)$, respectively. The independence of S_n and $\frac{\sum_{i=1}^m (X_i)^2}{(S_n)^2}$ is equivalent to

$$(3.6) \quad \phi(t, s) = \phi_1(t) \cdot \phi_2(s).$$

The left hand side of (3.6) becomes

$$\phi(t, s) = \int_0^\infty \cdots \int_0^\infty \exp \left[it(S_n) + \frac{is(\sum_{i=1}^m (X_i)^2)}{(S_n)^2} \right] dF(x_1) \cdots dF(x_n).$$

Also the right hand side of (3.6) becomes

$$\begin{aligned} \phi_1(t) \cdot \phi_2(s) &= \int_0^\infty \cdots \int_0^\infty \exp[it(S_n)] dF(x_1) \cdots dF(x_n) \\ &\cdot \int_0^\infty \cdots \int_0^\infty \exp\left[\frac{is(\sum_{i=1}^m (X_i)^2)}{(S_n)^2}\right] dF(x_1) \cdots dF(x_n). \end{aligned}$$

Then (3.6) gives

$$\begin{aligned} &\int_0^\infty \cdots \int_0^\infty \exp\left[it(S_n) + \frac{is(\sum_{i=1}^m (X_i)^2)}{(S_n)^2}\right] dF(x_1) \cdots dF(x_n) \\ (3.7) \quad &= \int_0^\infty \cdots \int_0^\infty \exp[it(S_n)] dF(x_1) \cdots dF(x_n) \\ &\cdot \int_0^\infty \cdots \int_0^\infty \exp\left[\frac{is(\sum_{i=1}^m (X_i)^2)}{(S_n)^2}\right] dF(x_1) \cdots dF(x_n). \end{aligned}$$

The integrals in (3.7) exist not only for reals t and s but also for complex values $t = u + iv$, $s = u^* + iv^*$, where u and u^* are reals, for which $v = \text{Im}(t) \geq 0$, $v^* = \text{Im}(s) \geq 0$ and they are analytic for all t , s for $v = \text{Im}(t) > 0$, $v^* = \text{Im}(s) > 0$, [see, Lukacs(1955)].

Differentiating (3.7) two times with respect to t and then one time respect to s and setting $s = 0$, we get

$$\begin{aligned} &\int_0^\infty \cdots \int_0^\infty (\sum_{i=1}^m (X_i)^2) \exp[it(S_n)] dF(x_1) \cdots dF(x_n) \\ (3.8) \quad &= \theta \int_0^\infty \cdots \int_0^\infty (S_n)^2 \exp[it(S_n)] dF(x_1) \cdots dF(x_n) \end{aligned}$$

where

$$\theta = E\left[\left(\frac{\sum_{i=1}^m (X_i)^2}{(S_n)^2}\right)\right].$$

The random variable θ is bounded and the moments exist. Then we know that

$$\begin{aligned} \theta &= E\left[\left(\frac{(X_2)^2 + \cdots + (X_{m+1})^2}{(S_n)^2}\right)\right] = E\left[\left(\frac{(X_1)^2 + (X_3)^2 + \cdots + (X_{m+1})^2}{(S_n)^2}\right)\right] \\ &= \cdots = E\left[\left(\frac{(X_{n-m+1})^2 + \cdots + (X_n)^2}{(S_n)^2}\right)\right] \end{aligned}$$

for i.i.d. random variables X_1, \dots, X_n . Then

$$(3.9) \quad {}_n C_m \theta = E\left[\left(\frac{{}^{n-1} C_{m-1} \sum_{i=1}^n X_i^2}{(S_n)^2}\right)\right] = E\left[\left(\frac{{}^{n-1} C_{m-1}}{1 + \frac{2\sum_{1 \leq i < j \leq n} X_i X_j}{\sum_{i=1}^n X_i^2}}\right)\right]$$

Note that, for $x_1, \dots, x_n > 0$, the relation $0 < 2\sum_{1 \leq i < j \leq n} x_i x_j \leq (n-1)(x_1^2 + \dots + x_n^2)$ holds and the equality on the right hand side occurs only if $x_1 = \dots = x_n$. By the assumed continuity of $F(x)$ we find $P(x_1 = \dots = x_n) = 0$ and $0 < \frac{2\sum_{1 \leq i < j \leq n} x_i x_j}{x_1^2 + \dots + x_n^2} < n-1$. It follows that $\frac{m}{n^2} < \theta < \frac{m}{n}$ by (3.9).

Let $\varphi(t)$ be the characteristic function of $F(x)$. Then

$$\varphi'(t) = i \int_0^\infty x \exp[itx] dF(x), \varphi''(t) = - \int_0^\infty x^2 \exp[itx] dF(x).$$

Expressing (3.8) as a differential equation for the characteristic function $\varphi(t)$, we get

$$m\varphi''(t)(\varphi(t))^{n-1} = \theta[n\varphi''(t)(\varphi(t))^{n-1} + 2 \cdot {}_n C_2 (\varphi'(t))^2 (\varphi(t))^{n-2}].$$

That is,

$$\frac{\varphi''(t)}{\varphi'(t)} = \left(\frac{2 \cdot {}_n C_2 \cdot \theta}{m - n\theta} \right) \frac{\varphi'(t)}{\varphi(t)}, \quad \frac{m}{n^2} < \theta < \frac{m}{n}.$$

After integrating with the initial conditions $\varphi(0) = 1, \varphi'(0) = iE(X)$, we get

$$(3.10) \quad \varphi'(t) = iE(X)(\varphi(t))^{\frac{2 \cdot {}_n C_2 \cdot \theta}{m - n\theta}}, \quad \frac{2 \cdot {}_n C_2 \cdot \theta}{m - n\theta} > 1.$$

The solution of this differential equation (3.10) with the above initial conditions is

$$\varphi(t) = \left(1 - \frac{iE(X)}{\lambda} t\right)^{-\lambda}, \quad \lambda = \frac{m - n\theta}{n^2\theta - m} > 0.$$

Consequently, $F(x)$ is a gamma distribution. □

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